$\longrightarrow u \longrightarrow j \longrightarrow z$

14.11. Representation of Graphs

Although a diagrammatic representation of a graph is very convenient for a visual study but this is only possible when the number of vertices and edges is reasonably small. Two types of representation are given below.

Matrix Representation. The matrix are commonly used to represent graphs for computer processing. The advantages of representing the graph in matrix form lies on the fact that many results of matrix algebra can be readily applied to study the structural properties of graphs from an algebraic point of view. There are number of matrices which one can associate with any graph. We shall discuss adjacency matrix and the incidence matrix.

Adjacency Matrix

(*a*) Representation of Undirected Graph

The adjacency matrix of an undirected graph G with n vertices and no parallel edges is an n by n matrix $A = [a_{ij}]$ whose elements are given by

$a_{ij} = 1$, if there is an edge between ith and jth vertices, and

$= 0$, if there is no edge between them.

Observations

(*i*) A is symmetric *i.e.* $a_{ij} = a_{ji}$ for all i and j.

(*ii*) The entries along the principal diagonal of A all 0's if and only if the graph has no self loops. A self loop at the vertex corresponds to $a_{ii} = 1$.

(*iii*) If the graph is simple (no self loop, no parallel edges), the degree of vertex equals the number of 1's in the corresponding row or column of A.

(*iv*) The (i, j) entry of A^m is the number of paths of length (no. of occurence of edges) m from vertex v_i to vertex v_j.

(*v*) If G be a graph with n vertices $v_1, v_2 \ldots\ldots\ldots\ldots v_n$ and let A denote the adjacency matrix of G with respect to this listing of the vertices. Let B be the matrix.

$$B = A + A^2 + A^3 + \ldots\ldots\ldots\ldots + A^n \; (n > 1)$$

Then G is a connected graph iff B has no zero entries.

This result can be used to check the connectedness of a graph by using its adjacency matrix.

Adjacency can also be used to represent undirected graphs with loops and multiple edges. A loop at the vertex v_i must have the element a_{ii} equal to 1 in the adjacency matrix. When multiple edges are present, the adjacency matrix is no longer a zero - one matrix, since the (i, j) th entry equals the number of edges that are associated between v_i and v_j. All undirected graphs, including multigraphs and pseudographs, have symmetric adjacency matrices.

(b) Representation of Directed Graph

The adjacency matrix of a diagraph D, with n vertices is the matrix $A = [a_{ij}]_{n \times n}$ in which

$$a_{ij} = 1 \text{ if arc } (v_i, v_j) \text{ is in } D$$
$$= 0 \text{ otherwise.}$$

Observations

(*i*) A is not necessary symmetric, since there may not be an edge from v_i to v_j when there is an edge from v_j to v_i

(*ii*) The sum of any column of j of A is equal to the number of arcs directed towards v_j.

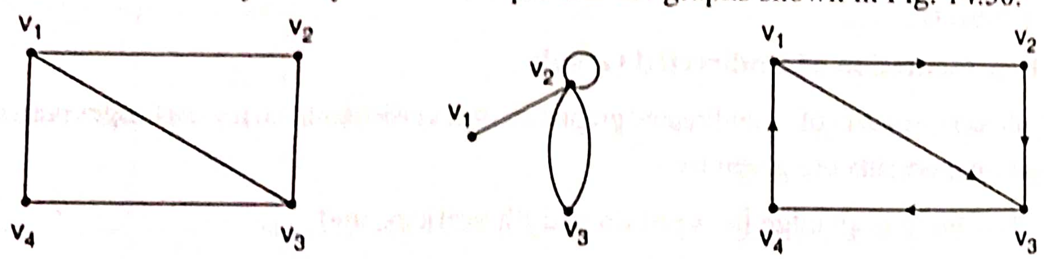
(*iii*) The sum of entries in row i is equal to the number of arcs directed away from vertex v_i (out degree of vertex v_i)

(*iv*) The (i, j) entry of A^m is equal to the number of path of length m from vertex v_i to vertex v_j.

(*v*) The diagonal elements of $A \cdot A^T$ show that out degree of the vertices. The diagonal entries of $A^T \cdot A$ shows the in degree of the vertices.

The adjacency matrices can also be used to represent directed multigraphs. Again such matrices are not zero – one matrices when there are multiple edges in the same direction connecting two vertices. In the adjacency matrix for a directed multigraph, a_{ij} equals the number of edges that are associated to (v_i, v_j).

Example 38 : Use adjacency matrix to represent the graphs shown in Fig. 14.50.



(a) (b) (c)

Fig. 14.50

Solution. We order the vertices in Fig 14.51 (a) as v_1, v_2, v_3 and v_4. Since there are four vertices, , the adjacency matrix representing the graph will be a square matrix of order four. The required adjacency matrix A is

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

We order the vertices in Fig. 14.51 (*b*) as v_1, v_2 and v_3. The adjacency matrix representing the graph with loop and multiple edges is given by

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix} \end{array}$$

Taking the order of the vertices in Fig. 14.51 (c) as v_1, v_2, v_3 and v_4. The adjacency matrix representing the digraph is given by

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example 39 : Draw the undirected graph represented by adjacency matrix A given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution. Since the given matrix is a square of order 5, the graph G has five vertices v_1, v_2, v_3, v_4 and v_5. Draw an edge from v_i to v_j where $a_{ij} = 1$. The required graph is drawn in Fig. 14.52.

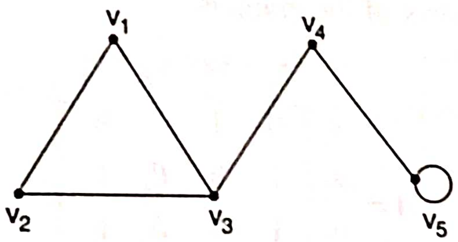


Fig. 14.51

Example 40: Draw the digraph G corresponding to adjacency matrix.

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Solution. Since the given matrix is square matrix of order four, the graph G has 4 vertices v_1, v_2, v_3 and v_4. Draw an edge from v_i to v_j where $a_{ij} = 1$. The required graph is shown in Fig.14.53.

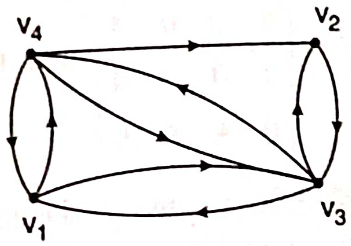


Fig. 14.52

Example 41: Draw the undirected graph G corresponding to adjacency matrix.

$$A=\begin{bmatrix}1 & 2 & 0 & 0\\ 3 & 0 & 1 & 1\\ 0 & 1 & 2 & 2\\ 0 & 1 & 2 & 0\end{bmatrix}$$

Solution. Since the given adjacency matrix is square matrix of order 4, G has four vertices v_1, v_2, v_3 and v_4. The matrix is not a zero-one matrix. Draw n edges from v_i to v_j where $a_{ij} = n$. Also, draw n loop at v_i where $a_{ij} = n$. The required graph is shown in Fig.14.53.

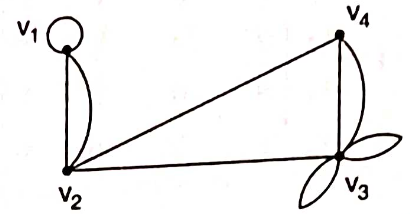


Fig. 14.53

Example 42. Consider the graph shown in Fig. 14.55. Find the number of walks of length three from v_2 to v_4 and also check the connectedness of the graph.

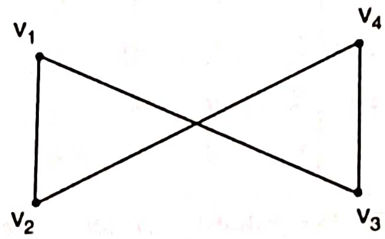


Fig. 14.54

Solution. The adjacency matrix of the graph is

$$A=\begin{array}{c} \\ v_1\\ v_2\\ v_3\\ v_4\end{array}\begin{array}{c}\begin{array}{cccc}v_1 & v_2 & v_3 & v_4\end{array}\\ \begin{bmatrix}0 & 1 & 1 & 0\\ 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\end{bmatrix}\end{array}$$

Now $$A^2=\begin{bmatrix}0 & 1 & 1 & 0\\ 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 1 & 0\\ 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\end{bmatrix}=\begin{bmatrix}2 & 0 & 0 & 2\\ 0 & 2 & 2 & 0\\ 0 & 2 & 2 & 0\\ 2 & 0 & 0 & 2\end{bmatrix}$$

Similarly, using matrix multiplication, we get

$$A^3=\begin{bmatrix}0 & 4 & 4 & 0\\ 4 & 0 & 0 & 4\\ 4 & 0 & 0 & 4\\ 0 & 4 & 4 & 0\end{bmatrix} \text{ and } A^4=\begin{bmatrix}8 & 0 & 0 & 8\\ 0 & 8 & 8 & 0\\ 0 & 8 & 8 & 0\\ 8 & 0 & 0 & 8\end{bmatrix}$$

Now the number of walks of length 3 from v_2 to v_4 = the element in the (2, 4) th entry in $A^3 = 4$. The four different edge sequence are $v_2-v_1-v_3-v_4$, $v_2-v_1-v_2-v_4$, $v_2-v_4-v_2-v_4$, and $v_2-v_4-v_3-v_4$

Again $B = A + A^2 + A^3 + A^4$

$$\therefore \quad B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 4 & 0 \\ 4 & 0 & 0 & 4 \\ 4 & 0 & 0 & 4 \\ 0 & 4 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

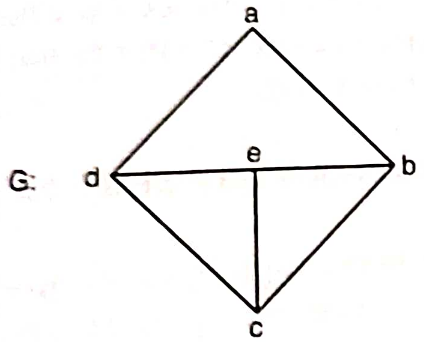
or
$$B = \begin{bmatrix} 10 & 5 & 5 & 10 \\ 5 & 10 & 10 & 5 \\ 5 & 10 & 10 & 5 \\ 10 & 5 & 5 & 10 \end{bmatrix}$$ Which contains no zero element.

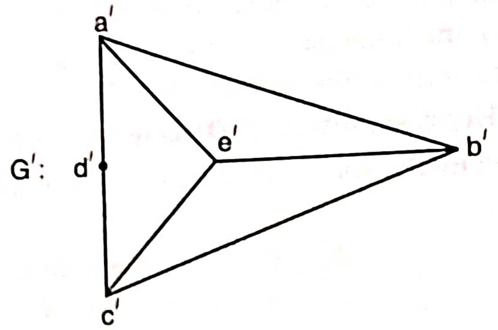
Hence the graph is a connected graph.

Note that one can use the adjacency matrix to check whether or not the given graphs G and G' are isomorphic. Two graphs are isomorphic if and only if their vertices can be labeled in such a way that the corresponding adjacency matrices are equal.

Theorem 14.18. Two graphs G_1 and G_2 are isomorphic if and only if the adjacency matrix of one is obtained from that of the other by interchanging rows and also columns in the same way.

Example 43. Show that the graphs G and G' are isomorphic.





Solution. Consider the map $f: G \to d'$, define as $f(a) = d', f(b) = a', f(c) = a', f(d) = c'$, and $f(e) = e'$. The adjacency matrix of G for the ordering a, b, c, d and e is

$$A(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The adjacency matrix of G′ for the ordering d', a', b', c' and e' is

$$A(G') = \begin{array}{c} \\ d' \\ a' \\ b' \\ c' \\ e' \end{array} \begin{array}{c} \begin{array}{ccccc} d' & a' & b' & c' & e' \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

i.e. $A(G) = A(G')$

$\therefore$ G and G' are isomorphic.

Incidence Matrix

(a) Representation of Undirected Graph

Consider a undirected graph $G = (V, E)$ which has n vertices and m edges all labelled. The incidence matrix $I(G) = [b_{ij}]$, is then $n \times m$ matrix, where

$$b_{ij} = 1 \quad \text{when edge } e_j \text{ is incident with } v_i$$
$$= 0 \quad \text{otherwise}$$

Observations

(i) Each column of B comprises exactly two unit entries.

(ii) A row with all 0 entries corresponds to an isolated vertex.

(iii) A row with a single unit entry corresponds to a pendant vertex.

(iv) The number of unit entries in row i of B is equal to the degree of the corresponding vertex v_i

(v) The permutation of any two rows (any two columns) of $I(G)$ corresponds to a relabelling of the vertices (edges) of G.

(vi) Two graphs are isomorphic if and only if their corresponding incidence matrices differ only by a permutation of rows and columns.

(vii) If G is connected with n vertices then the rank of $I(G)$ is $n - 1$.

Incidence matrices can also be used to represent multiple edges and loops. Multiple edges are represented in the incidence matrix using columns with identical entries. Since these edges are incident with the same pair of vertices. Loops are represented using a column with exactly one entry equal to l, corresponding to the vertex that is incident with this loop.

(b) Representation of Directed Graph

The incidence matrix $I(D) = [b_{ij}]$ of digraph D with n vertices and m edges is the $n \times m$ matrix in which.

$$b_{ij} = 1 \text{ if arc } j \text{ is directed away from } a \text{ vertex } v_i$$
$$= -1 \text{ if arc } j \text{ is directed towards vertex } v_i$$
$$= 0 \text{ otherwise.}$$

Example 44 : Find the incidence matrix to represent the graph shown in Fig.14.55.

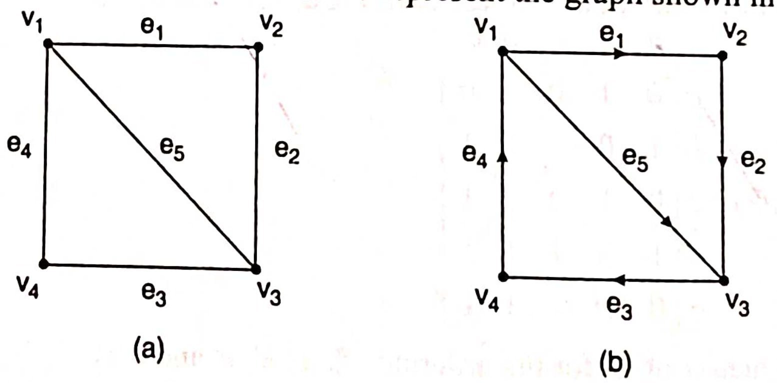


Fig. 14.55

Solution. The incidence matrix of Fig.(a) is obtained by entering for row v and column e is 1 if e is incident on v and 0 otherwise. The incidence matrix is.

$$I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The incidence matrix of the digraph of Fig.(*b*) is

$$I(D) = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

Example 45. Draw the graph whose incidence matrix is

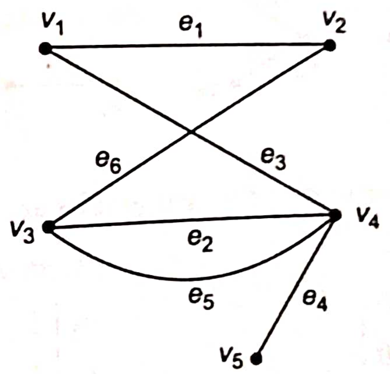
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Solution. Since the given matrix has 5 rows and 6 columns, its corresponding graph has 5 vertices and 6 edges. We re-write the incidence matrix as follows:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Since first column contains two 1 placed at 2nd and 3rd row, vertices v_2 and v_3 are connected by e_1. Similarly, the vertices v_1 and v_4 are connected by e_2, and the vertices v_2 and v_4 are connected by e_3 and so on.

Therefore, the graph of the given incidence matrix is



Example 46. Draw the graph whose incidence matrix is given below:

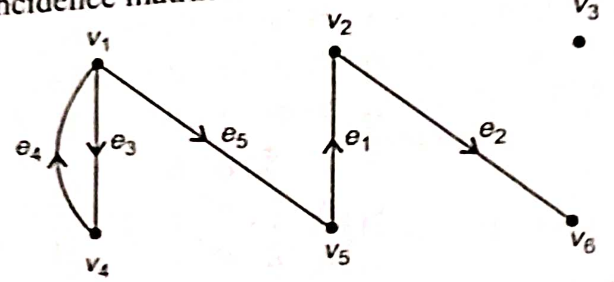
$$\begin{bmatrix} 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

Solution. Since the given matrix has 6 rows and 5 columns, its corresponding graph has 6 vertices and 5 edges. Since the matrix has the element – 1, it is a digraph. We rewrite the incidence matrix as follows:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \end{array}$$

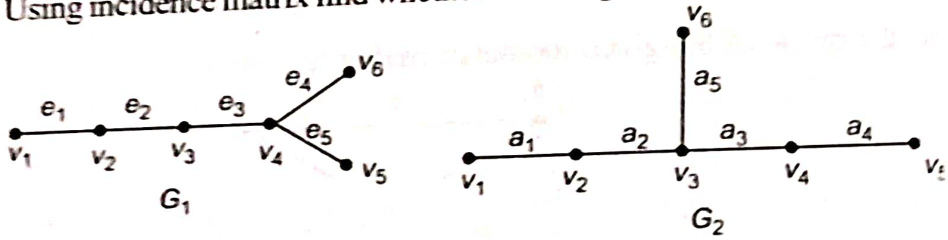
Since first column contains -1 placed at 2nd row, and 1 placed at 4th row, the edge e_1 is directed from v_4 to v_2. The edge e_2 has no direction and it is connected by v_2 and v_5, the edge e_3 has direction from v_1 to v_6, the edge e_4 from v_6 to v_1, the edge e_5 from v_1 to v_4 but the vertex v_3 is not connected to any vertex. Hence v_3, is isolated vertex and the graph is not connected.

Hence, the graph of the incidence matrix is



Theorem 14.19. Two graphs G_1 and G_2 are isomorphic if and only if the incidence matrix of one is obtained from that of the other by permutation of rows/columns of the matrix.

Example 47. Using incidence matrix find whether the two given graphs G_1 and G_2 are isomorphic.



Solution. The incidence matrices of the two graphs are

$$I(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad I(G_2) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array}\begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Now, if $I(G_2)$ can be obtained from $I(G_1)$ by interchanging rows and columns, then G_1 and G_2 are isomorphic.

We see 1st three columns of $I(G_1)$ and $I(G_2)$ are same. To make the fourth column identical we interchange 5th row and 6th row of $I(G_1)$.

and get a matrix $B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

We see first four columns of $I(G_2)$ and B are same. The only fifth column of B is not same as that of $I(G_2)$.

So, we conclude $I\ (G_2)$ can not be obtained from $I\ (G_1)$ by interchanging rows and columns. Hence the two graphs G_1 and G_2 are not isomorphic.